RAYLEIGH'S METHOD

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Continuous system

The potential energy: $$T_{\max} = \frac{1}{2} Z_0^2 \omega^2 \int_0^L m(x) \, [\psi(x)]^2 \, dx$$

The kinetic energy : $$V_{\max} = \frac{1}{2} Z_0^2 \int_0^L EI(x) \, [\psi''(x)]^2 \, dx$$

After equating the maximum potential energy to the maximum kinetic energy,the squared frequency is found to be

Since $V_{\max} = T_{\max}$ for a conservative system, we get

$$\boxed{\omega^2 = \frac{\int_0^L EI(x) \, [\psi''(x)]^2 \, dx}{\int_0^L m(x) \, [\psi(x)]^2 \, dx}}$$

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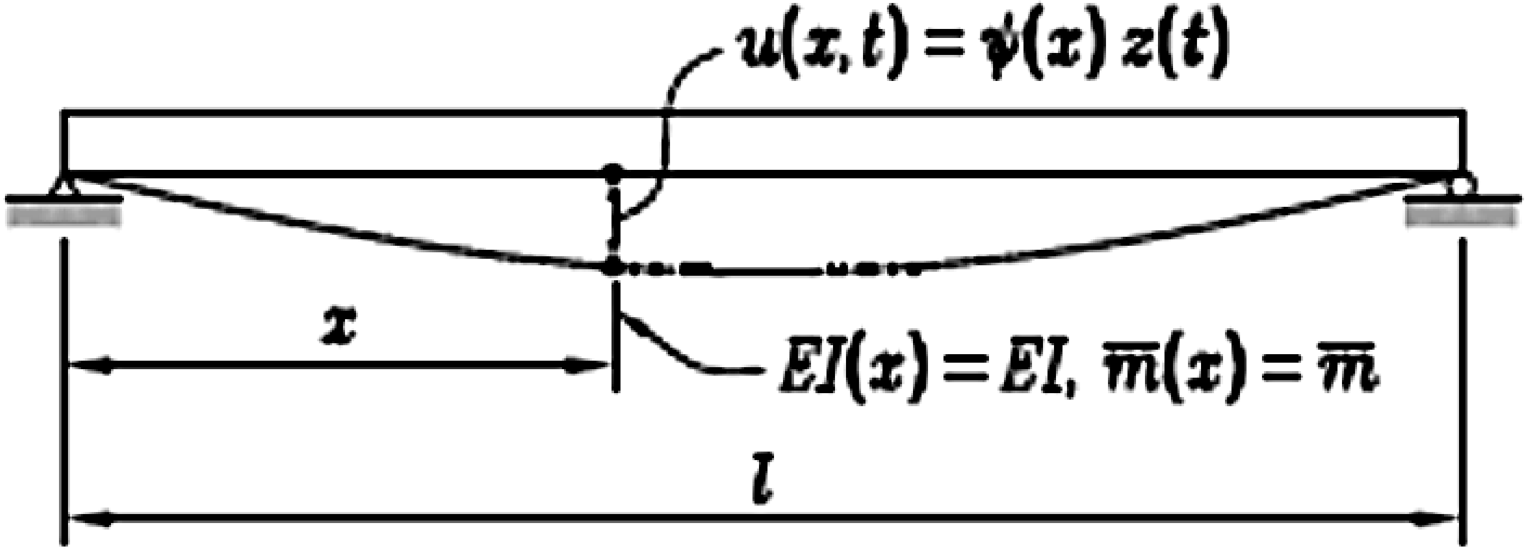
Problem 1:

Calculate the fundamental frequency of the simply supported beam shown in Figure. The beam has constant stiffness *EI* and constant linear mass $\bar{m}$.



Simply supported uniform beam

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Solution:

We first assume a parabolic displacement function that is expressed as

$$\psi(x) = \frac{x}{l}\left(\frac{x}{l} - 1\right)$$

and which satisfies the boundary conditions ψ(x = 0) = ψ(x = l) = 0. The second derivative of the function with respect to x is

$$\psi''(x) = \frac{2}{l^2}.$$

The maximum strain energy is

$$V_{\max} = \frac{1}{2} z_o^2 \int_0^l EI(x)\left(\psi''(x)\right)^2 \mathrm{d}x = \frac{1}{2} z_o^2 EI \int_0^l \left(\frac{2}{l^2}\right)^2 \mathrm{d}x = \frac{1}{2} z_o^2 \frac{4EI}{l^3}.$$

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We get the maximum kinetic energy

$$T_{max} = \frac{1}{2} z_o^2 \omega^2 \int_0^l \bar{m}(x) \left(\psi(x)\right)^2 \mathrm{d}x$$

$$= \frac{1}{2} z_o^2 \omega^2 \bar{m} \int_0^l \left[\frac{x}{l}\left(\frac{x}{l} - 1\right)\right]^2 \mathrm{d}x = \frac{1}{2} z_o^2 \omega^2 \frac{\bar{m} l}{30}.$$

Writing $V_{max} = T_{max}$, we find according to equation

$$\omega^2 = \frac{\int_0^l EI(x)\left(\psi''(x)\right)^2 \mathrm{d}x}{\int_0^l \bar{m}(x)\left(\psi(x)\right)^2 \mathrm{d}x} = \frac{120EI}{\bar{m} l^4}$$

from which

$$\omega = 10.9545 \sqrt{\frac{EI}{\bar{m} l^4}}.$$

2. We now assume a sinusoidal displacement function, expressed as

$$\psi(x) = \sin\left(\frac{\pi x}{l}\right)$$

which also satisfies the boundary conditions ψ(x = 0) = ψ(x = l) = 0. The second
derivative of this function with respect to x is

$$\psi''(x) = -\frac{\pi^2}{l^2}\sin\left(\frac{\pi x}{l}\right)$$

we get the maximum deformation energy

$$V_{\max} = \frac{1}{2}z_o^2\int_0^l EI(x)\left(\psi''(x)\right)^2 \mathrm{d}x = \frac{1}{2}z_o^2 EI\frac{\pi^4}{l^4}\int_0^l \sin^2\left(\frac{\pi x}{l}\right)\mathrm{d}x = \frac{1}{2}z_o^2\frac{\pi^4}{2l^3}EI.$$

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We get the maximum kinetic energy

$$T_{max} = \frac{1}{2} z_o^2 \omega^2 \int_0^l \bar{m}(x)\left(\psi(x)\right)^2 \mathrm{d}x = \frac{1}{2} z_o^2 \omega^2 \bar{m} \int_0^l \sin^2\left(\frac{\pi x}{l}\right) \mathrm{d}x = \frac{1}{2} z_o^2 \omega^2 \frac{\bar{m} l}{2}.$$

Writing Vmax = Tmax, we find according to equation

$$\omega^2 = \frac{\int_0^l EI(x)\left(\psi''(x)\right)^2 \mathrm{d}x}{\int_0^l \bar{m}(x)\left(\psi(x)\right)^2 \mathrm{d}x} = \pi^4 \frac{EI}{\bar{m} l^4} = 97.4091 \frac{EI}{\bar{m} l^4}$$

from which

$$\omega = 9.8696 \sqrt{\frac{EI}{\bar{m} l^4}}.$$

Note:

The sinusoidal displacement function resulted in a frequency, ω, that is 11% lower than the value obtained with the parabolic function.

It is interesting to note that the selected sinusoidal function is actually the exact deformation shape of the first vibration mode of a simply supported beam and the calculated frequency is the exact frequency.

Problem 2:

Calculate the natural frequency of the simply supported uniform beam in previous example . using a displacement function equal to the deformed shape of the beam subjected to

(1) a uniformly distributed load equal to the unit weight of the beam and

(2) a concentrated load equal to the total weight of the beam applied at mid span.

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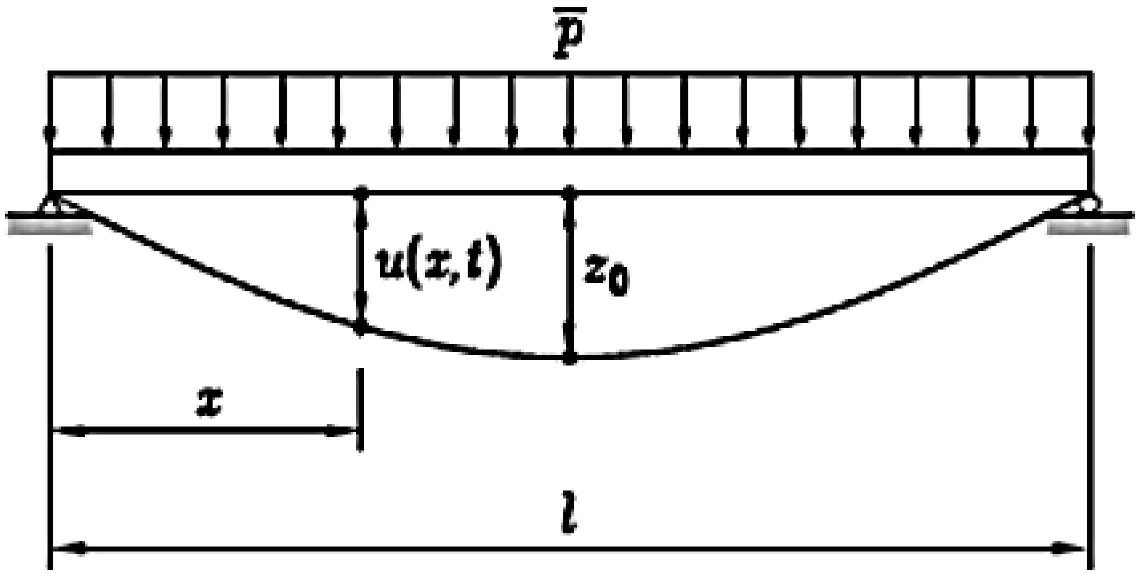
Solution:

1. We assume a displacement function equal to the deformed shape of the beam under a uniformly distributed load $\bar{p} = \bar{m}g$ as shown in Figure.

$$\psi(x) = \frac{16z_o}{5l^4}(l^3x - 2lx^3 + x^4)$$



Deformed shape of a uniform simply supported beam subjected to a uniformly distributed load where zo = 5 $\bar{p}$ l4/(384EI). ψ(x) satisfies the boundary conditions

ψ(x = 0) =ψ(x = l) = 0.

The second derivative of the function with respect to x is

$$\psi''(x) = -\frac{192}{5} z_o \frac{x(l-x)}{l^4}.$$

we get the maximum deformation energy

$$V_{\max} = \frac{1}{2} z_o^2 \int_0^l EI(x) \left(\psi''(x)\right)^2 \mathrm{d}x$$

$$= \frac{1}{2} z_o^2 EI \int_0^l \left(-\frac{192}{5} z_o \frac{x(l-x)}{l^4}\right)^2 \mathrm{d}x = \frac{3072}{125} \frac{z_o^4 EI}{l^3}.$$

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We get the maximum kinetic energy

$$T_{max} = \frac{1}{2} z_o^2 \omega^2 \int_0^l \bar{m}(x)(\psi(x))^2 \mathrm{d}x$$

$$= \frac{1}{2} z_o^2 \omega^2 \bar{m} \int_0^l \left(\frac{16 z_o}{5 l^4} (l^3 x - 2 l x^3 + x^4) \right)^2 \mathrm{d}x = \frac{1984}{7875} z_o^4 \omega^2 \bar{m} l.$$

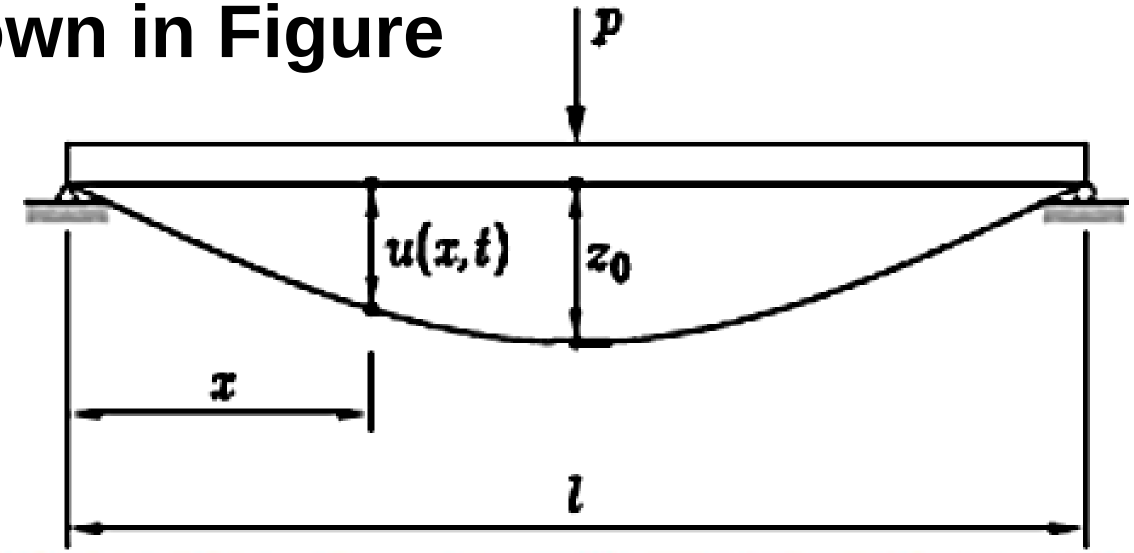
Writing $V_{max} = T_{max}$, we find according to equation

$$\omega^2 = \frac{3024}{31} \frac{EI}{\bar{m} l^4} = 97.5 \frac{EI}{\bar{m} l^4}$$

from which

$$\omega = 9.8767 \sqrt{\frac{EI}{\bar{m} l^4}}.$$

2. We assume a displacement function equal to the deformed shape of the beam under a concentrated load p = mg applied at mid span as shown in Figure



Deformed shape of a uniform simply supported beam subjected to a concentrated load applied at midspan

We will use the following displacement functions:

$$\psi(x) = \begin{cases} \frac{z_o}{l^3}(3l^2x - 4x^3) & \text{for } 0 \le x \le l/2 \\ \frac{z_o}{l^3}(-l^3 + 9l^2x - 12lx^2 + 4x^3) & \text{for } l/2 \le x \le l \end{cases}$$

where zo = pl3/(48EI). ψ(x) satisfies the boundary conditions
ψ(x = 0) = ψ(x =l) = 0.
The second derivatives of these functions are

$$\psi''(x) = \begin{cases} -\frac{24z_o}{l^3}x & \text{for } 0 \le x \le l/2 \\ -\frac{24z_o}{l^3}(l-x) & \text{for } l/2 \le x \le l. \end{cases}$$

we get the maximum deformation energy

$$\mathcal{V}_{\max} = \frac{1}{2}z_o^2 \int_0^l EI(x)\left(\psi''(x)\right)^2 \mathrm{d}x$$

$$= \frac{1}{2}z_o^2 EI \left(\int_0^{l/2} \left[-\frac{24z_o}{l^3}x \right]^2 \mathrm{d}x + \int_{l/2}^{l} \left[-\frac{24z_o}{l^3}(l-x) \right]^2 \mathrm{d}x \right) = 24\frac{z_o^4 EI}{l^3}.$$

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We get the maximum kinetic energy

$$\text{Tmax} = \frac{1}{2} z_o^2 \omega^2 \int_0^l \bar{m}(x)(\psi(x))^2 \mathrm{d}x$$

$$= \frac{1}{2} z_o^2 \omega^2 \bar{m} \left(\int_0^{l/2} \left[\frac{z_o}{l^3} (3l^2 x - 4x^3) \right]^2 \mathrm{d}x \right.$$

$$\left. + \int_{l/2}^{l} \left[\frac{z_o}{l^3} (-l^3 + 9l^2 x - 12lx^2 + 4x^3) \right]^2 \mathrm{d}x \right) = \frac{17}{70} z_o^4 \omega^2 \bar{m} l.$$

Writing Vmax = Tmax, we find according to equation

$$\omega^2 = \frac{1680}{17} \frac{EI}{\bar{m} l^4} = 98.8 \frac{EI}{\bar{m} l^4}$$

from which

$$\omega = 9.9410 \sqrt{\frac{EI}{\bar{m} l^4}}.$$

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The same result would be obtained if the integration of the assumed deformed shape was carried out twice between the limits 0 ≤ l ≤ l/2 – a much easier solution – since the function is symmetric with respect to a vertical axis passing at l/2.
The frequency calculated for an assumed deformed shape corresponding to a uniformly distributed load is less than the one obtained for an assumed deformed shape corresponding to a uniformly distributed load at mid span and is, therefore, a better approximation of the true fundamental frequency.
The natural frequencies of the uniform beam in problem 1and 2 canbe written as

$$\omega = \alpha\sqrt{EI/\bar{m}l^4}.$$

Estimations of the natural frequency of a simply supported uniform beam using the Rayleigh method are presented in Table 1, as a function of parameter α, for different displacement functions. The results illustrate the property of the Rayleigh quotient, which states that frequency values obtained with a displacement function that is different from the exact deformed shape of the fundamental mode are always greater than the exact frequency.

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Example	$\psi(x)$	α	Error, %
proplem1:	$\frac{x}{l}\left(\frac{x}{l}-1\right)$	10.9545	10.99
proplem2:	$\frac{z_o}{l^3}\left(3l^2x-4x^3\right), \quad 0 \le x \le l/2$	9.9410	0.72
	$\frac{z_o}{l^3}\left(-l^3+9l^2x-12lx^2+4x^3\right), \quad l/2 \le x \le l$		
proplem2:	$\frac{16z_o}{5l^4}\left(l^3x-2lx^3+x^4\right)$	9.8767	0.07
proplem1:	$\sin\left(\frac{\pi x}{l}\right)$	9.8696	0.00

Table 1: *Estimates of the natural frequency of a simply supported uniform beam with the Rayleigh method*